UNIT 13 : UNIFORM CONVERGENCE OF SEQUENCES AND SERIES

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13.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- learn about uniform convergence of sequence and series
- know about the Cauchy's general principle of uniform convergence
- determine the tests for uniform convergence
- learn about relation between Uniform convergence and continuity
- learn about relation between Uniform convergence and integration
- learn about relation between Uniform convergence and differentiation.

13.2 INTRODUCTION

In the previous units, we learned about the sequences and series. Usually, we considered those sequences and series whose terms are numbers. In this chapter, we shall study about those sequences and series

whose terms depend ona variable i.e. the terms are real-valued functions defined on an interval as domain. The terms of such sequences and series are denoted by $f_n(x)$ and the sequences and series are of the form $\{f_n\}$ and $\sum f_n$ respectively.

13.3 POINTWISE CONVERGENCE

We start the section by defining point-wise limit of the sequence $\{f_n\}$ and series $\sum f_n$. Later in the section, we will encounter new category of problems which arise when sequences (viz. series) of variable terms and the limit function f of the terms f_n are considered. We investigate under what supplementary conditions, the properties of the terms f_n are transferred to the limit function f. Henceforth, in this respect, in the next section, we introduce the concept of uniform convergence of sequence (viz. series) in the domain of definition.

Let $\{f_n\}$ be a sequence of functions, defined on an interval I : = [a, b]. For each point $x_1 \in [a,b]$, there corresponds a sequence of numbers $\{f_n(x_1)\}$ with terms

$$f_1(x_1), f_2(x_1), f_3(x_1).....$$

We suppose that the sequence of numbers $\{f_n(x_1)\}$ converges for every $x_1 \in [a,b]$. Let $\{f_n\}$ converges to $f(x_1)$.

Similarly, let the sequences at points x_1, x_2, x_3 ..... of I converges to

$f(x_1), f(x_2), f(x_3).....$

Definition : A real-valued function f with domain I : = [a, b] and range set defined by (1) is said to be the limit or point-wise limit of the sequence $\{f_n\}$ on [a, b] and the sequence $\{f_n\}$ is said to be pointwise convergent to f on [a, b] if

$$f(x) = \lim_{n\to\infty} \{f_n(x)\},\ \forall x \in I$$

Similarly, if the series $\sum f_n$ converges for every point $x \in I$, we define

$$f(x) = \sum_{n=0}^{\infty} f_n(x),\ \forall x \in [a,b]$$

as the sum or the pointwise sum of the series $x \in I$ on I : = [a, b]

Definition : f is said to be the point-wise limit of the sequence of functions $\{f_n\}$ defined on [a, b], if for each $\in > 0$, and for each $x \in [a, b]$, there exists an integer m such that

$$\left|f_n(x) - f(x)\right| < \in, \ \forall\, n \geq m$$

Now, a natural question arises that for a sequence (viz, series) of variable terms, whether properties belonging to terms viz. boundedness, continuity, integrability, differentiablility etc. are transferred to the limit function of the corresponding sequence (viz, series). The example below will give the answers.

Example 1 : The geometric series

$1 + x + x^2 + ...$

converges to $(1 - x)^{-1}$ in the interval $-1 < x < 1$.

All the terms $f_1(x) - 1,\ f_2(x) = x^2, ...$ are bounded but the sum $\sum_{n=0}^{\infty} f_n(x) = (1-x)^{-1}$ is not bounded.

Example 2 : We consider the series $\sum_{n=0}^{\infty} f_n$ where

$$f_n(x) = \frac{x^2}{(1+x^2)^n}$$

and x is real At x = 0, each $f_n(x) = 0$, so the sum of the series f (0) == 0.

For $x \neq 0$, $\sum_{n=0}^{\infty} f_n$ is a geometric series with common ration $\frac{1}{1+x^2}$, and thus $f(x) = 1 + x^2$

Hence,

$$f(x) = \begin{cases} 1+x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Thus each term of the series is continuous but the sum f is not continuous.

Example 3 : The sequence $\{f_n\}$ where $f_n = \frac{\sin nx}{\sqrt{n}}$ and x real, has the limit

$$f(x) = \lim_{n \to \infty} f_n(x) = 0$$

Therefore, $f'(x) = 0$ and so $f'(0) = 0$

But $f_n'(0) = \sqrt{n} \cos nx$

and so $f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$

Thus at x = 0, the sequences $\{f_n'(x)\}$ diverges whereas the limit $f'(x) = 0$ i.e. the limit of differentials is not equal to the differential of the limit.

Example 4 : The sequences $\{f_n\}$ where $f_n(x) = nx(1-x^2)^n$ for $0 \le x \le 1,\ n = 1, 2, \ldots$ has the limit

$$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

At x = 0, each $f_n(0) = 0$, so that $\lim_{n\to\infty} f_n(0) = 0$

Thus the limit function $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for $0 \le x \le 1$.

Therefore

$$\int_0^1 f(x)dx = 0$$

Again

$$\int_0^1 f_n(x)dx = \int_0^1 nx(1-x^2)^n dx = \frac{n}{2n+2}$$

so that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \Big\} = 1/2$$

Thus

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \Big\} \neq \int_0^1 f(x)\,dx = \int_0^1 |\{f_n\}|dx$$

Thus the limit of integral is not equal to the integral of the limit. In other words, the sequence of integrals maynot converge to the integral of the limit of the sequence.

At the end of this section, we can conclude that a new categoryof problem arises with the consideration of sequences (viz. series) of variable terms. Thus to determine the conditions so that the properties of the terms f_n are transferred to the limit function f, we introduce the concept of Uniform convergence of sequences (viz. series) in the next chapter.

13.4 UNIFORM CONVERGENCE

Definition : A sequence of functions $\{f_n\}$ is said to be converge uniformly to a function f on an interval [a, b], if for any $\in > 0$, and for all $x \in [a, b]$, there exists an integer N (independent of x but dependent on $\in$) such that for all

$x \in [a, b]$,

$$\left| f_n(x) - f(x) \right| < \in, \ \forall n \geq N$$

From the definition, it is clear that every uniformly convergent sequence is pointwise convergent and the uniform limit function is same as the pointwise limit function. However the converse is not always true.

Definition : A series of function $\sum f_n$ is said to converge uniformly on [a, b] if the sequence $\{S_n\}$ of its partial sums, defined as

$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on [a, b]

13.5 CAUCHY'S GENERAL PRINCIPLE OF UNIFORM CONVERGENCE

Theorem : A sequence of functions $\{f_n\}$ defined on [a, b] converges uniformly on [a, b] if and only if for every $\in > 0$ and for all $x \in [a, b]$, there exists an integer N such that

$$\left| f_{n+p}(x) - f_n(x) \right| < \in, \ \forall n \geq N, p \geq 1$$

Proof : Necessary Part : Let the sequence $\{f_n\}$ uniformly converges on [a, b] to the limit function F i.e. for given $\in > 0$ and for all $x \in [a, b]$, there exists an integers m_1 such that

$$\left| f_n(x) - f(x) \right| < \in, \ \forall n \geq m_1$$

Similarly for the same $\in > 0$ and for all $x \in [a, b]$, there exists an integers m_2 such that

$$\left| f_{n+p}(x) - f(x) \right| < \in / 2, \ \forall n \geq m_2, p \geq 1$$

Let N = max $\{m_1, m_2\}$

Therefore

$$\left| f_{n+p}(x) - f_n(x) \right| \leq \left| f_{n+p}(x) - f(x) \right| + \left| f_n(x) - f(x) \right|$$
$$< \in / 2 + \in / 2 = \in, \ \forall n \geq N, \ p \geq 1$$

Sufficient part : Let for $\in > 1$ and for all $x \in [a, b]$, there exists an integers N such that

$$\left| f_{n+p}(x) - f_n(x) \right| < \in, \ \forall n \geq N, p \geq 1$$

By Cauchy's general principle of converge (stated in chapter 3), $\{f_n\}$ converges for each $x \in [a, b]$ to a limit, say f. Thus the sequence converges pointwise t f.

We now prove the convergence is uniform.

For a given $\in > 0$. let us choose an integer N such that (2) holds. We fix n and let $p \to \infty$ in (2). Since $f_{n+p} \to f$ as $p \to \infty$, we get

$$\left| f(x) - f_n(x) \right| < \in, \forall\, n \geq N,\ \forall\, x \in [a, b]$$

which proves that $f_n(x) \to f(x)$ uniformly on [a, b].

Remark : In the statement of the theorem, (2) may be equivalently replaced by

$$\left| f_n(x) - f_m(x) \right| < \in, \forall\, n, m \geq N$$

Example : Test the uniform convergence of the sequence $\{f_n\}$ where

$$f_n(x) = \frac{nx}{1 + n^2 x^2},\ \forall\, x \in R$$

The sequence converges pointwise to f where

$$f(x) = 0,\ \forall\, x \in R$$

Let $\{f_n\}$ converges uniformly in any interval [a, b] so that the point-wise limit is also the uniform limit. Therefore for given $\in > 0$, there exists an N such that $\forall\, x \in [a, b]$

$$\left| \left| \frac{nx}{1 + n^2 x^2} - 0 \right| < \in \forall n \geq N \right|$$

Taking $\in = 1/3$ and inteer m > N such that $1/m \in$ [a, b], we find n = m and x = 1/m that

$$\frac{nx}{1 + n^2 x^2} = \frac{1}{2} < \frac{1}{3} = \in$$

Thus, we arrive at a contradiction and so the sequence is not uniformly convergent in the interval [a, b] containing the point 1/m. But since $1/m \to 0$, the interval [a, b] contains 0. Hence, the sequence is not uniformly convergent on any interval [a, b] containing 0.



CHECK YOUR PROGRESS 1

Q 1: Show that the sequence $\{f_n\}$ such that

$$f_n(x) = \frac{1}{x+n}$$

is uniformly convergent in any interval [0, b].

Q 2: Show that the series $\sum f_n$ whose sun to n terms $S_n(x) = nxe^{-nx2}$ is pointwise and not uniformly convergent on any interval [0, k].

13.6 TEST FOR UNIFORM CONVERGENCE

In the previous section, we got introduced with the concept of uniform convergence. We have used merely the definition of uniform convergence for the purpose of determining whether the sequence (viz. series) is uniformly converging in a given interval. In this section, we will inquire how we can determine whether a given sequence or a series does or doest ot converge uniformly in a given interval.

Theorem 1 : Let $\{f_n\}$ be a sequence of functions such that

$$\lim_{n\to\infty} \{f_n(x)\} = f(x),\ x \in [a,b]$$

and let $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ Then $f_n \to f$ uniformly on [a, b] if and only if $M_n \to 0$ as $n \to \infty$

Proof : Necessary part : Let $f_n \to f$ uniformly on [a, b] so that for a given $\in > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \in,\ \forall n \geq N,\ Ax \in [a,b]$$

$$\Rightarrow M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \in,\ An \geq N$$

$$\Rightarrow M_n \to 0 \text{ as } n \to \infty$$

Sufficient part : Let $M_n \to 0$ as $n \to \infty$ so that for any $\in > 0$, there exists an integer N such that

$$M_n < \in,\ \forall n \geq N$$

$$\Rightarrow \sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \in,\ A\, n \geq N$$

$$\Rightarrow |f_n(x) - f(x)| < \in, \ \forall n \geq N, \ Ax \in [a, b]$$

$$\Rightarrow f_n \to f \text{ uniformly on [a, b]}$$

Theorem 2 : Weierstrass's M-test A series of functions $\sum f_n$ will converge uniformly on [a, b] if there exists a convergent series $\sum f_n$ of positive numbers such that for all $x \in [a, b]$

$$|f_n(x)| \leq M_n, \ \forall n$$

Proof : Let $\in > 0$ be a positive number.

Since $\sum M_n$ is convergent, therefore there exists a positive integer N such that

$$|M_{n+1} + M_{n+2} + ... + M_{n+p}| < \in, \ \forall n \geq N, \ p \geq 1$$

Hence, for all $x \in [a, b]$ and for all $n \geq N, \ p \geq 1$, we have

$$|f_{n+1}(x) + f_{n+2}(x) + ... + f_{n+p}(x)| \leq |f_{n+1}(x)| + ... + |f_{n+p}(x)| \qquad (3)$$

$$M_{n+1} + M_{n+2} + ...M_{n+p} < \in \qquad (4)$$

Equatiosn (3) and (4) imply that $\sum f_n$ is uniformly convergent on [a, b].

Remark : The converse of the theorem 2 is not true in general.

Besides the Weierstrass's M-test, we have two other tests viz. Abel's and Dirichlet's test to test the uniform convergence of a series.

Theorem 3 : Abel's test If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of $x \in [a, b]$ and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\sum u_n(x)$ is uniformly convergent on [a, b], then so also is the series $\sum b_n(x) u_n(x)$.

Theorem 4 : Dirichlet's test If $b_n(x)$ is a monotonic function of n for each fixed value of $x \in [a, b]$ and $b_n(x)$ tends uniformly to zero for $a \leq x \leq b$ and if there is a number K > 0 independent of x and n, such that for all values of $x \in [a, b]$,

$$\left| \sum_{r=1}^{n} u_r(x) \right| \leq K, \forall n$$

then the series $\sum b_n(x) u_n(x)$ is uniformly convergent on [a, b]

Theorem : (A) If a sequence $\{f_n\}$ converges uniformly in [a, b] and x_0 is a point of [a, b] such that

$$\lim_{x\to x_0} f_n(x) = a_n,\ n = 1, 2, \ldots.$$

then $\{a_n\}$ converges and $\lim_{x\to x_0} f(x) = \lim_{x\to\infty} a_n$

(B) If a series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f in [a, b] and x_0 is a point in [a, b] such that

$$\lim_{x\to x_0} f_n(x) = a_n,\ n = 1, 2, \ldots.$$

then $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{x\to x_0} f(x) = \lim_{x\to\infty} a_n$

We now look at few examples to test the uniform convergence of sequence and series.

Example : Show that the sequence $\{f_n\}$ such that

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

is not uniformly convergent on any interval containing 0.

We have $\lim_{n\to\infty} f_n(x) = 0$ for all x.

Now $\frac{nx}{1+n^2x^2}$ attains maximum value 1/2 at x = 1/n. And $1/n \to 0$ as $n \to \infty$. Taking an interval [a, b] containing 0, we have

$$M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$$

$$= \sup \left| \frac{nx}{1+n^2x^2} \right| = 1/2$$

which does not tend to 0 as $n \to \infty$

Hence the sequence $\{f_n\}$ is not uniformly convergent in any interval containing 0.



CHECK YOUR PROGRESS 2

Q 1: Prove that the sequence $\{f_n\}$ where

$$f_n(x) = \frac{x}{1+nx^2}$$

converges uniformly on any closed interval I.

13.7 UNIFORM CONVERGENCE AND CONTINUITY

In this section, we present theorems connecting continuity and uniform convergence of series (viz. sequences).

Theorem : (A) If $\{f_n\}$ is a sequence of continuous functions on an interval [a, b] and if $f_n \to f$ uniformly on [a, b], then f is continuous on [a, b].

(B) If a series $\sum f_n$ converges uniformly to f in an [a, b] and its terms f_n are continuous at a point x_0 of the interval, then the sum function f is also continuous at x_0.

Proof : (B) Since $\sum f_n$ converges uniformly to f on [a, b], therefore for $\in > 0$, we can choose N such that for all $x \in [a, b]$

$$\left| \sum_{r=1}^{N} f_r(x) - f(x) \right| < \in / 3, \ \forall n \geq N$$

and in particular, at a point $x_0 \in [a, b]$ and n = N

$$\left| \sum_{r=1}^{N} f_r(x_0) - f(x_0) \right| < \in / 3$$

Again, since each f_n is continuous at x_0, the sum of a finite number of functions $\sum_{r=1}^{N} f_r$ is also continuous at $x = x_0$. Therefore, for $\in > 0$, there exists $\delta > 0$ such that

$$\left| \sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} f_r(x_0) \right| < \in / 3 \text{ for } |x - x_0| < \delta$$

Hence for $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \leq \left| f(x) - \sum_{r=1}^{N} f_r(x) \right| + \left| \sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} f_r(x_0) \right|$$

$$+ \left| \sum_{r=1}^{N} f_r(x_0) - f(x_0) \right|$$

$$\in / 3 + \in / 3 + \in / 3$$

which implies $f(x) \to f(x_0)$ when $x \to x_0$ i.e. the sum function f is continuous at $x = x_0$.

Corollary : If a series $\sum f_n$ converges uniformly to f on an interval and if the functions f_n are all continuous throughout the interval, then so is the sum function f.

Proof : The proof follows from the fact that if x_0 is an arbitrary point of [a, b], then the above theorem holds for all points of [a, b].

Remark : The converse of the theorem may not be true i.e. series (viz. sequence) of continuous terms exist which have a continuous sum (limit) but are not uniformly convergent. In other words, the condition of uniform convergence is only sufficient, but not necessary.

Theorem : Dini's theorem on uniform convergence (A) If a sequence of continuous functions $\{f_n\}$, defined on [a, b] is monotonic increasing, and converges (pointwise) to a continuous function f, then the convergence is uniform on [a, b].

(B) If the sum function of a series $\sum f_n$ with non-negative continuous terms defined on an interval [a, b] is continuous on [a, b], then the series is uniformly convergent on the interval.

Example : Show that the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots$$

is not uniformly convergent on [0, 1].

The terms of the series are continuous and the series converges pointwise to f where $f(x) = \begin{cases} 1+x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$ which is discontinuous on [0, 1].

Hence the series does not converge uniformly on [0, 1].



CHECK YOUR PROGRESS 3

Q 1: Show that the series $\sum \frac{x}{(nx+1)\{(n-1)x+1\}}$ is uniformly convergent on any interval [a, b] but not pointwise convergent on [0, b] where 0 < a < b.

13.8 UNIFORM CONVERGENCE AND INTEGRATION

In this section, we present theorem connecting the integrability and uniform convergence of sequence (viz.) series.

Theorem : (A) If a sequence $\{f_n\}$ converges uniformly to f on [a, b], and each function f_n is integrable, then f is integrable on [a, b], and the sequence $\left\{\int_a^x f_n dt\right\}$ converges uniformly to $\left\{\int_a^x f dt\right\}$ on [a, b] i.e.

$$\int_a^x f dt = \lim_{n\to\infty} \int_a^x f_n dt, \ \forall x \in [a,b]$$

(B) If a series $\sum f_n$ converges uniformly to f on [a, b], and each term $f_n(x)$ is integrable, then f is integrable on [a, b], and the series $\sum \left(\int_a^x f_n dt\right)$ converges uniformly to $\int_a^x f dt$ on [a, b] i.e.

$$\int_a^x f dt = \sum_{n=1}^{\infty} \left(\int_a^x f_n dt\right), \ \forall x \in [a,b]$$

We then say that the series is integrable term by term.

Proof : (A) Let $\in > 0$ be any number.

By the uniform convergence of the sequence, there exists an integer m such that for all $x \in [a,b]$ such that

$$|f_n(x) - f(x)| < \in /3(b-a), \ \forall n \geq m$$

In particular,

$$|f_m(x) - f(x)| < \in /3(b-a)$$

For this fixed m, since f_m is integrable, we choose a partition P of [a, b], such that

$$U(P, f_m) - L(P, f_m) < \in /3$$

From (5), we have

$$f(x) < f_m(x) + \in /3(b-a)$$

$$\Rightarrow U(P,f) < U(P,f_m) + \in /3$$

Again from (5), we have

$$f(x) < f_m(x) + \in /3(b-a)$$

$$\Rightarrow L(P,f) < L(P,f_m) - \in /3$$

From (6), (7) and (8), we get

$$U(P,f)-L(P,f)<U(P,f_m)-L(P,f_m)+2\in/3$$
$$<\in/3+\in/3=\in$$

Thus f is integrable on [a, b]

We now prove that the sequence $\left\{\int_a^x f_n dt\right\}$ coverges uniformly to $\left\{\int_a^x f dt\right\}$ on [a, b] i.e.

$$\int_a^x f dt=\lim_{n\to\infty}\int_a^x f_n dt,\ \forall x\in[a,b]$$

Since the sequence $\{f_n\}$ converges uniformly to f, therefore for $\in>0$, there exists an integer N such that for all $x\in[a,b]$

$$\left|f_n(x)-f(x)\right|<\in/3(b-a),\ \forall n\geq N$$

Then for all $x\in[a,b]$ and $n\geq N$, we have

$$\left|\int_a^x f dt-\int_a^x f_n dt\right|=\left|\int_a^x (f-f_n)dt\right|\leq\int_a^x |f-fn|dt$$

This implies $\left\{\int_a^x f_n dt\right\}$ converges uniformly to $\left\{\int_a^x f dt\right\}$ on [a, b] i.e.

$$\int_a^x f dt=\lim_{n\to\infty}\int_a^x f_n dt,\ \forall x\in[a,b]$$

13.9 UNIFORM CONVERGENCE AND DIFFERENTIATION

In this section, we present theorem connecting the differentiability and uniform convergence of sequence (viz.) series.

Theorem : Let $\{f_n\}$ be a sequence of differentiable functions on [a, b] such that it converges at least at one point $x_0\in[a,b]$. If the sequence of differentials $\left\{f_n'\right\}$ converges uniformly to G on [a, b], then the given sequence $\{f_n\}$ converges uniformly on [a, b] to f and $'(x) = G(x)$.

(B) Let $\sum f_n$ be a series of differentiable functions on [a, b] such that it converges at least at one point $x_0\in[a,b]$. If the series of differentials $\sum f_n$ converges uniformly on [a, b] to f and $f'(x) = G(x)$.

Proof : Let $\in>0$ be any number.

By the convergence of $\{f_n(x_0)\}$ and the uniform convergence of $\{f_n'\}$ for $\in > 0$, we can choose a positive integer N such that for all $x \in [a,b]$,

$$|f_{n+p}(x_0) - f_n(x_0)| < \in/2,\ \forall n \geq N,\ p \geq 1 \quad (9)$$

$$|f'_{n+p}(x) - f'_n(x)| < \in/2(b-a),\ \forall n \geq N,\ p \geq 1 \quad (10)$$

Applying Lagrange's mean value theorem to the function ... for any two points x and t of [a, b], we get for $x < \xi < t$, for all $n \geq N,\ p \geq 1$, for all $n \geq N,\ p \geq 1$.

$$|f_{n+p}(x) - f_n(x) - f_{n+p}(t) + f_n(t)| = |f'_{n+p}(\xi) - f'_n(\xi)| \quad (11)$$

$$< \frac{|x-t|}{2(b-a)} \quad (12)$$

$$< \in/2 \quad (13)$$

and

$$|f_{n+p}(x) - f_n(x)| \leq |f_{n+p}(x) - f_n(x) - f_{n+p}(t) + f_n(t)| + |f_{n+p}(x) - f_n(x)|$$

$< \in/2 + \in/2 = \in$ using (12) and (13)

Thus, the sequences $\{f_n\}$ uniformly converges on [a, b]

Let it converge to f.

Now, for a fixed x on [a, b] and for $t \in |a,b|,\ t \neq x$, let us define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x},\ n = 1,2,.....$$

and

$$\phi(t) = \frac{f(t) - f(x)}{t-x}$$

Since each f_n is differentiable, therefore for each n,

$$\lim_{t \to x} \phi_n(x) = f'_n(x)$$

Thus

$$|\phi_{n+p}(x) - \phi_n(x)| = \frac{1}{|t-x|} |f_{n+p}(t) - f_{n+p}(x) - f_n(t) + f_n(x)|$$

$$= \frac{1}{|t-x|} \{|f_{n+p}(t) - f_n(t) - f_{n+p}(x) - f_n(x)|\}$$

$$< \frac{\in}{2(b-a)},\ \forall n \geq N,\ p \geq 1 \quad \text{Using (12)}$$

so that $\{\phi_n(t)\}$ converges uniformly on [a, b] for $t \neq x$.

Since $\{f_n\}$ also converges uniformly on f, therefore

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t - x} = \frac{f(t) - f(x)}{t = x} = \phi(t)$$

Thus $\{\phi_n(t)\}$ converges uniformly on [a, b] for $t \neq x$.

Also, form (14), we get

$$\lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x) = G(x)$$

Thus exists and therefore f is differentiable and $\lim_{t\to x} \phi(t) = f'(x)$.

Hence $f'(x) = G(x) = \lim_{n\to\infty} f_n'(x)$ which completes the proof.

We state the simple version of the theorem for series, that for sequence too is on the same line.

Theorem : Let a series $\sum f_n$ of differentiable functions converges pointwise to f on [a, b] and each f_n' is continuous on [a, b], and the series $\sum f_n'$ converges uniformly to G on [a, b], then the given series $\sum f_n$ converges uniformly to f on [a, b], and $f'(x) = G(x)$.



13.10 LET US SUM UP

- A real-valued function f with domain I : [a, b] and range set defined by 91) is said to be the limit or point-wise limit of the sequence $\{f_n\}$ on [a, b] and the sequence $\{f_n\}$ is said to be pointwise convergent to f on [a, b] if

  $$f(x) = \lim_{n\to\infty} \{f_n(x)\}, \forall x \in I$$

- Similarly, if the series $\sum f_n$ converges for every point $x \in I$, we define

  $$f(x) = \sum_{n=0}^{\infty} f_n(x), \quad \forall x \in [a, b]$$

  as the sum or the pointwise sum of the series $\sum f_n$ on I : = [a, b]

- f is said to be the point-wise limit of the sequence of functions $\{f_n\}$ defined on [a, b], if for each $\in > 0$, and for each $x \in [a, b]$, there exists an integer m such that

  $$|f_n(x) - f(x)| < \in, \forall n \geq m$$

- A sequence of functions $\{f_n\}$ is said to be converge uniformly to a

function f on an interval [a, b], if for any $\in > 0$, and fr all $x \in [a,b]$, there exists an integer N (independent of x but dependent on $\in$) such that for all $x \in [a,b]$,

$$\left| f_n(x) - f(x) \right| < \in, \ \forall n \geq N$$

- Every uniformly convergent sequence is pointwise convergent and the uniform limit function is same as the pointwise limit function. However the converse is not always true.
- A series of function $\sum f_n$ is said to converge uniformly on [a, b] if the sequence $\{S_n\}$ of its partial sum, defined as

$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on [a, b].

- A sequence of functions $\{f_n\}$ defined on [a, b] converges uniformly on [a, b] if and only if for every $\in > 0$ and for all $x \in [a,b]$, there exists an integer N such that

$$\left| f_{n+p}(x) - f_n(x) \right| < \in, \ \forall n \geq N, \ p \geq 1$$

- Let $\{f_n\}$ be a sequence of functions such that

$$\lim_{n \to \infty} \{f_n(x)\} = f(x), \ x \in [a,b]$$

and let $M_n = \text{sul}_{x \in [a,b]} \left| f_n(x) - f(x) \right|$ Then $f_n \to f$ uniformly on [a, b] if and only if $M_n \to 0$ as $n \to \infty$

- Weierstrass's M-test : A series of functions $\sum f_n$ will converge uniformly on [a, b] if there exists a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a,b]$

$$\left| f_n(x) \right| \leq M_n, \ \forall n$$

- Abel's test : If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of $x \in [a,b]$ and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\sum u_n(x)$ is uniformly convergent on [a, b], then so also is the series $\sum b_n(x) u_n(x)$.
- If a sequence $\{f_n\}$ converges uniformly in [a, b] and x_0 is a point of [a, b] such that

$$\lim_{x \to x0} f_n(x) = a_n, \ n = 1, 2, ...$$

then $\{a_n\}$ converges and $\lim_{x\to x_0} f_n(x) = \lim_{x\to\infty} a_n$

- If a series converges uniformly to f in [a, b] and x_0 is a point in [a, b] such that

$$\lim_{x\to x_0} f_n(x) = a_n,\ n = 1, 2, ...$$

then $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{x\to x_0} f_n(x) = \lim_{x\to\infty} a_n$

- Dirichlet's test : If $b_n(x)$ is a monotonic function of n for each fixed value of $x \in [a, b]$ and $b_n(x)$ tends uniformly to zero for $a \le x \le b$ and if there is a number K > 0 independent of x and n, such that for all values of $x \in [a, b]$,

$$\sum_{r=1}^{n} u_r(x) < K,\ \forall n$$

then the series $\sum b_n(x)u_n(x)$ is uniformly convergent on [a, b]

- If $\{f_n\}$ is a sequence of continuous functions on an interval [a, b], and if $f_n \to f$ uniformly on [a, b], then f is continuous on [a, b].
- If a series $\sum f_n$ converges uniformly to f in an [a, b] and its terms f_n are continuous at a point x_0 of the interval, then the sum function f is also continuous at x_0.
- Dini's theorem on uniform convergence : If a sequence of continuous functions $\{f_n\}$ defined on [a, b] is monotonic increasing, and converges (pointwise) to a continuous function f, then the convergence is uniform on [a, b].
- If the sum function of a series $\sum f_n$ with non-negative continuous terms defined on an interval [a, b] is continuous on [a, b], then the series is uniformly convergent on the interval.
- If a sequence $\{f_n\}$ converges uniformly to f on [a, b] and each function f_n is integrable, then f is integrable on [a, b], and the sequence $\left\{\int_a^x f_n dt\right\}$ converges uniformly to $\left\{\int_a^x f dt\right\}$ on [a, b] i.e.

$$\int_a^x f dt = \lim_{n\to\infty} \int_a^x f_n dt,\ \forall x \in [a, b]$$

- If a series $\sum f_n$ converges uniformly to f on [a, b], and each term $f_n(x)$

is integrable, then f is integrable on [a, b], and the series $\sum\left(\int_a^x f_n dt\right)$ converges uniformly to $\int_a^x f dt$ on [a, b] i.e.

$$\int_a^x f dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right), \forall x \in [a,b]$$

- Let $\{f_n\}$ be a sequence of differentiable functions on [a, b] such that it converges at least at one point $x_0 \in [a,b]$. If the sequence of differentials $\{f_n'\}$ converges uniformly to G on [a, b], then the given sequence $\{f_n\}$ converges uniformly on [a, b] fo f and $'x = G(x)$.
- Let $\sum f_n$ be a series of differentiable functions on [a, b] such that it converges at least at one point $x_0 \in [a,b]$. If the series of differentials $\sum f_n'$ converges uniformly to G on [a, b], then the given series $\sum f_n$ converges uniformly on [a, b] to f and $f'(x) = G(x)$.
- Let a series $\sum f_n$ of differentiable functions converges pointwise to f on [a, b] and each f_n' is continuous on [a, b], and the series $\sum f_n'$ converges uniformly to G on [a, b], then the given series $\sum f_n$ converges uniformly to f on [a, b], and $f'(x) = G(x)$.



13.11 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1 :

Ans to Q No 1: We have

$$f(x) = \lim_{n\to\infty} f_n(x) = 0, \ \forall x \in [0,b]$$

and hence the sequence converges pointwise to 0.

Now, for any $\in > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \in$$

if $n > (1/\in) - x$, which decreases with x, the maximum value being $1/\in$.

Let N be an integer greater than or equal to $1/\in$, so that for $\in > 0$, there exists N such

$$\left|f_n(x)-f(x)\right|<\in,\ \forall\ n\geq N$$

Hence the sequence converges uniformly in [0, b].

Ans to Q No 2: Try Yourself.

Check your Progress 2

Ans to Q No 1: Try Yourself.

Check your Progress 3

Ans to Q No 1: Try Yourself.



13.12 FURTHER READING

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13.13 MODEL QUESTIONS

Q 1: Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent for all real x.

Q 2: Show that the series $\sum \frac{(-1)^n}{n} |x|^n$ is uniformly convergent in $-1 \le x \le 1$.

Q 3: Show that $\sum a_n / n^x$ is uniformly convergent in [0, 1] if $\sum a_n$ converges.

Q 4: Prove that the series $\sum x^n / n^2$ is uniformly convergent in [–1, 1].

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